

Pauli equation for joint tomographic probability distribution of spin 1/2 particle

Ya. A. Korennoy, V. I. Man'ko

*P.N. Lebedev Physics Institute,
Leninskii prospect 53, 119991, Moscow, Russia*

Abstract

The positive vector optical tomogram fully describing the quantum state of spin 1/2 particle without any redundancy is introduced. Reciprocally the vector symplectic tomogram and vector quasidistributions $\vec{W}(\mathbf{q}, \mathbf{p})$, $\vec{Q}(\mathbf{q}, \mathbf{p})$, $\vec{P}(\vec{\alpha})$ are introduced. The evolution equations for proposed vector optical and symplectic tomograms and vector quasidistributions for arbitrary Hamiltonian are obtained. The quantum system of charged spin 1/2 particle in arbitrary electro-magnetic field is considered in proposed representations and evolution equations which are analogs of Pauli equation are obtained. The propagator of evolution equation in the case of homogeneous and stationary magnetic field in Landau gauge is found and the evolution of initial entangled superposition of lower Landau levels in the vector optical representation is considered. The system of linear quantum oscillator with spin in vector optical tomography representation is considered and the evolution of initial entangled superposition of two lower Fock states and spin-up, spin-down states is studied in this representation.

Keywords: Pauli equation, evolution equation, quantum tomography, optical tomogram of quantum state, vector-portrait of state, spin tomogram, tomographic probability, Landau levels.

1 Introduction

The tomographic approach [1, 2] to the quantum state of a system has allowed one to establish a map between the density operator (or any its representation) and a set of probability distributions, often called ‘quantum tomograms’. The latter have all the characteristics of classical probabilities; they are non-negative, measurable and normalized.

Based on this connection, a classical-like description of quantum dynamics by means of ‘symplectic tomography’ has been formulated [3, 4], providing a bridge between classical and quantum worlds. The tomographic distribution for rotated spin variables has been constructed in [5], and the same approach has been followed in [6]. Different aspects of classical-like description using tomographic probabilities were given in [7, 8, 9].

The main deficiency of the proposed spin tomograms is a redundancy of information. Attempts to reduce or avoid such a redundancy were made in [10, 11, 12, 13, 14, 15, 16, 17]. The spin tomography was also studied in [18, 19] and in other papers.

The tomographic formulation of quantum evolution equation was suggested in [20] for symplectic tomograms. For optical tomograms it was given in [21, 23]. The first attempt of foundation of the Pauli equation in tomographic representation was done in [24]. Using the version of spin tomogram with redundancy of information the authors obtained a complicated evolution equation.

The aim of our work is the consideration of the special case of spin 1/2 particle quantum state tomography without redundancy of information, constructing the joint vector distribution for space coordinates and spin projections, and finally deriving the evolution equation for such distribution, which would be an analogue of the Pauli equation. It would also be a simplification of approach attempted in [24].

The paper is organized as follows. In Sec. 2 we give basic formulas of tomographic representation of quantum mechanics and the evolution equation for optical and symplectic tomogram of nonrelativistic spinless quantum system with arbitrary Hamiltonian. In Sec. 3 we introduce a positive four-component vector probability description of spin 1/2 particle and give the evolution equation for such a vector-portrait of quantum state with arbitrary Hamiltonian. In Sec. 4 charged spin 1/2 particle in arbitrary electro-magnetic field is considered in proposed representations and evolution equations which are analogs of Pauli equation are obtained. In Sec. 5 vector quasidistributions $\vec{W}(\mathbf{q}, \mathbf{p})$, $\vec{Q}(\mathbf{q}, \mathbf{p})$, $\vec{P}(\vec{\alpha})$ are introduced and analogs of Pauli equation for $\vec{W}(\mathbf{q}, \mathbf{p})$ and $\vec{Q}(\mathbf{q}, \mathbf{p})$ are obtained. In Sec. 6 the propagator of evolution equation in the case of homogeneous and stationary magnetic field in Landau gauge is found and the evolution of initial entangled superposition of lower Landau levels in the vector optical representation is considered. In Sec. 7 the system of linear quantum oscillator with spin in vector optical tomography representation is considered and the evolution of initial entangled superposition of two lower Fock states and spin-up, spin-down states is studied. The conclusion and prospects are presented in Sec. 8.

2 Evolution of spinless quantum systems in probability representation

Let us review the constructions of the optical and symplectic tomograms for spinless systems. The relationships between the density operator $\hat{\rho}$ and the optical tomogram $w(\vec{X}, \vec{\theta})$ of the system in the invariant form [25] are written as follows

$$w(\vec{X}, \vec{\theta}) = \text{Tr}\{\hat{\rho}\hat{U}_w(\vec{X}, \vec{\theta})\}, \quad \hat{\rho} = \int w(\vec{X}, \vec{\theta})\hat{D}_w(\vec{X}, \vec{\theta})d^n X d^n \theta, \quad (1)$$

where dequantizer $\hat{U}_w(\vec{X}, \vec{\theta})$ and quantizer $\hat{D}_w(\vec{X}, \vec{\theta})$ operators equal respectively

$$\hat{U}_w(\vec{X}, \vec{\theta}) = |\vec{X}, \vec{\theta}\rangle \langle \vec{X}, \vec{\theta}| = \hbar^{n/2} \prod_{\sigma=1}^n (m_\sigma \omega_{0\sigma})^{-n/2} \delta \left(X_\sigma \hat{1} - \hat{q}_\sigma \cos \theta_\sigma - \hat{p}_\sigma \frac{\sin \theta_\sigma}{m_\sigma \omega_{0\sigma}} \right), \quad (2)$$

$$\hat{D}_w(\vec{X}, \vec{\theta}) = \left(\frac{\sqrt{\hbar}}{2\pi} \right)^n \int \prod_{\sigma=1}^n \frac{|\eta|}{\sqrt{m_\sigma \omega_{0\sigma}}} \exp \left\{ i\eta_\sigma \left(X_\sigma - \hat{q}_\sigma \cos \theta_\sigma - \hat{p}_\sigma \frac{\sin \theta_\sigma}{m_\sigma \omega_{0\sigma}} \right) \right\} d^n \eta, \quad (3)$$

where $|\vec{X}, \vec{\theta}\rangle$ is an eigenfunction of the operator $\hat{X}(\vec{\theta})$ with components $\hat{X}_\sigma = \hat{q}_\sigma \cos \theta_\sigma + \hat{p}_\sigma \sin \theta_\sigma$ corresponding to the eigenvalue \vec{X} . Notion of quantizer and dequantizer is related to star product quantization schemes (see recent review [22]).

The von-Neumann equation without interaction with the environment

$$i\hbar \frac{\partial}{\partial t} \hat{\rho} = [\hat{H}, \hat{\rho}] \quad (4)$$

in the optical tomography representation has the form [23]

$$\partial_t w(\vec{X}, \vec{\theta}, t) = \frac{2}{\hbar} \int \text{Im} \left[\text{Tr} \left\{ \hat{H} \hat{D}(\vec{X}', \vec{\theta}') \hat{U}(\vec{X}, \vec{\theta}) \right\} \right] w(\vec{X}', \vec{\theta}', t) d^n X' d^n \theta', \quad (5)$$

and for a large class of Hamiltonians $\hat{H}(\hat{\mathbf{p}}, \hat{\mathbf{q}}, t)$, when the Hamiltonian is an analytic function of position $\hat{\mathbf{q}}$ and momentum $\hat{\mathbf{p}}$ components, it can be written as follows

$$\partial_t w(\vec{X}, \vec{\theta}, t) = \hat{\mathcal{M}}_w(\vec{X}, \vec{\theta}, t) w(\vec{X}, \vec{\theta}, t). \quad (6)$$

where the operator $\hat{\mathcal{M}}(\vec{X}, \vec{\theta}, t)$ is obtained from the Hamiltonian

$$\hat{\mathcal{M}}_w(\vec{X}, \vec{\theta}, t) = \frac{2}{\hbar} \text{Im} \hat{H} \left([\hat{\mathbf{p}}]_w(\vec{X}, \vec{\theta}), [\hat{\mathbf{q}}]_w(\vec{X}, \vec{\theta}), t \right),$$

which is an operator depending on two operators of position $\tilde{\mathbf{q}}$ and momentum $\tilde{\mathbf{p}}$ in the tomographic representation

$$[\hat{q}_\sigma]_w(\vec{X}, \vec{\theta}) = \sin \theta_\sigma \frac{\partial}{\partial \theta_\sigma} \left[\frac{\partial}{\partial X_\sigma} \right]^{-1} + X_\sigma \cos \theta_\sigma + i \frac{\hbar \sin \theta_\sigma}{2m_\sigma \omega_\sigma} \frac{\partial}{\partial X_\sigma}, \quad (7)$$

$$[\hat{p}_\sigma]_w(\vec{X}, \vec{\theta}) = m\omega_{0\sigma} \left(-\cos \theta_\sigma \left[\frac{\partial}{\partial X_\sigma} \right]^{-1} \frac{\partial}{\partial \theta_\sigma} + X_\sigma \sin \theta_\sigma \right) - \frac{i\hbar}{2} \cos \theta_\sigma \frac{\partial}{\partial X_\sigma}. \quad (8)$$

Similarly, for symplectic tomogram $M(X, \mu, \nu, t)$ one can be written

$$M(\vec{X}, \vec{\mu}, \vec{\nu}, t) = \text{Tr} \{ \hat{\rho} \hat{U}_M(\vec{X}, \vec{\mu}, \vec{\nu}) \}, \quad \hat{\rho} = \int M(\vec{X}, \vec{\mu}, \vec{\nu}, t) \hat{D}_M(\vec{X}, \vec{\mu}, \vec{\nu}) d^n X d^n \mu d^n \nu, \quad (9)$$

where dequantizer $\hat{U}_M(\vec{X}, \vec{\mu}, \vec{\nu})$ and quantizer $\hat{D}_M(\vec{X}, \vec{\mu}, \vec{\nu})$ operators are respectively equal

$$\hat{U}_M(\vec{X}, \vec{\mu}, \vec{\nu}) = |\vec{X}, \vec{\mu}, \vec{\nu}\rangle \langle \vec{X}, \vec{\mu}, \vec{\nu}| = \hbar^{n/2} \prod_{\sigma=1}^n (m_\sigma \omega_{0\sigma})^{-n/2} \delta(X_\sigma \hat{1} - \hat{q}_\sigma \mu_\sigma - \hat{p}_\sigma \nu_\sigma), \quad (10)$$

$$\hat{D}_M(\vec{X}, \vec{\mu}, \vec{\nu}) = \frac{1}{(2\pi\sqrt{\hbar})^n} \prod_{\sigma=1}^n (m_\sigma \omega_{0\sigma})^{3n/2} \exp \left\{ i \sqrt{\frac{m_\sigma \omega_{0\sigma}}{\hbar}} (X_\sigma - \hat{q}_\sigma \mu_\sigma - \hat{p}_\sigma \nu_\sigma) \right\}, \quad (11)$$

where $|\vec{X}, \vec{\mu}, \vec{\nu}\rangle$ is an eigenfunction of the operator $\hat{X}(\vec{\mu}, \vec{\nu})$ with components $\hat{X}_\sigma = \mu_\sigma \hat{q}_\sigma + \nu_\sigma \hat{p}_\sigma$ corresponding to the eigenvalue \vec{X} .

For the same Hamiltonians the evolution equation for the symplectic tomogram [23]

$$\partial_t M(\vec{X}, \vec{\mu}, \vec{\nu}, t) = \hat{\mathcal{M}}_M(\vec{X}, \vec{\mu}, \vec{\nu}, t) M(\vec{X}, \vec{\mu}, \vec{\nu}, t), \quad (12)$$

with notation

$$\hat{\mathcal{M}}_M(\vec{X}, \vec{\mu}, \vec{\nu}, t) = \frac{2}{\hbar} \text{Im} \hat{H} \left([\hat{\mathbf{p}}]_M(\vec{X}, \vec{\mu}, \vec{\nu}), [\hat{\mathbf{q}}]_M(\vec{X}, \vec{\mu}, \vec{\nu}), t \right),$$

where $[\hat{\mathbf{q}}]_M$ and $[\hat{\mathbf{p}}]_M$ are operators of positions and momentums in the symplectic representation

$$[\hat{p}_\sigma]_M = \left(- \left[\frac{\partial}{\partial X_\sigma} \right]^{-1} \frac{\partial}{\partial \nu_\sigma} - i \frac{\mu_\sigma \hbar}{2} \frac{\partial}{\partial X_\sigma} \right), \quad [\hat{q}_\sigma]_M = \left(- \left[\frac{\partial}{\partial X_\sigma} \right]^{-1} \frac{\partial}{\partial \mu_\sigma} + i \frac{\nu_\sigma \hbar}{2} \frac{\partial}{\partial X_\sigma} \right). \quad (13)$$

3 Probability description of spin 1/2 particle

As known, that pure states of quantum spin 1/2 particle are described by two-component spinor wave functions (ψ_1, ψ_2) , and mixed states can be described by density matrixes $\hat{\rho}_{ij}$, where $i, j = 1, 2$. In the case of pure state

$$\hat{\rho} = \begin{pmatrix} \psi_1^* \psi_1 & \psi_1^* \psi_2 \\ \psi_2^* \psi_1 & \psi_2^* \psi_2 \end{pmatrix}. \quad (14)$$

The density matrix satisfy Pauli equation (4) in the von-Neumann form with the 2×2 matrix operator Hamiltonian. The density matrix is a normalized nonnegative hermitian matrix and thus it is actually defined by four real scalar components. To construct our representation we should pick up four positive components (spatial distributions), which completely define the density matrix.

Let us choose such components as follows:

$$\begin{aligned} w_1(\vec{X}, \vec{\theta}, t) &= \text{Tr} \left\{ \hat{\rho}(t) \hat{U}_w(\vec{X}, \vec{\theta}) \otimes |s_1 = 1/2\rangle \langle s_1 = 1/2| \right\}, \\ w_2(\vec{X}, \vec{\theta}, t) &= \text{Tr} \left\{ \hat{\rho}(t) \hat{U}_w(\vec{X}, \vec{\theta}) \otimes |s_2 = 1/2\rangle \langle s_2 = 1/2| \right\}, \\ w_3(\vec{X}, \vec{\theta}, t) &= \text{Tr} \left\{ \hat{\rho}(t) \hat{U}_w(\vec{X}, \vec{\theta}) \otimes |s_3 = 1/2\rangle \langle s_3 = 1/2| \right\}, \\ w_4(\vec{X}, \vec{\theta}, t) &= \text{Tr} \left\{ \hat{\rho}(t) \hat{U}_w(\vec{X}, \vec{\theta}) \otimes |s_3 = -1/2\rangle \langle s_3 = -1/2| \right\}, \end{aligned} \quad (15)$$

where $\hat{U}_w(\vec{X}, \vec{\theta})$ is a spinless dequantizer operator (2) and $|s_j = \pm 1/2\rangle$ is an eigenfunction of the projection of spin operator to the direction q_j corresponding to the eigenvalue $\pm 1/2$. In more compact form formula

(15) can be written as follows

$$\vec{w}(\vec{X}, \vec{\theta}, t) = \text{Tr} \left\{ \hat{\rho}(t) \vec{U}_w(\vec{X}, \vec{\theta}) \right\}, \quad (16)$$

where $\vec{w}(\vec{X}, \vec{\theta}, t)$ is a four component vector of probability distributions and dequantizer operator $\vec{U}_w(\vec{X}, \vec{\theta})$ has the form

$$\vec{U}_w(\vec{X}, \vec{\theta}) = \hat{U}_w(\vec{X}, \vec{\theta}) \otimes \vec{\mathcal{U}}, \quad (17)$$

where $\vec{\mathcal{U}}$ is a four-component vector of 2×2 matrices

$$\vec{\mathcal{U}} = \left\{ \hat{\mathcal{U}}_{j(kl)} \right\} = \left(\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right). \quad (18)$$

Here the first index $j = 1, 2, 3, 4$ is the number of the component of the four-component vector, and (kl) are the indexes of 2×2 matrices. The inverse transform of $\vec{w} \rightarrow \hat{\rho}$ can be written in terms of quantizer operator $[\vec{D}_{jk}]_w(\vec{X}, \vec{\theta})$ as follows

$$\hat{\rho}_{jk}(t) = \int [\vec{D}_{jk}]_w(\vec{X}, \vec{\theta}) \vec{w}(\vec{X}, \vec{\theta}, t) d^3X d^3\theta, \quad j, k = 1, 2. \quad (19)$$

Quantizer operator $[\vec{D}_{jk}]_w(\vec{X}, \vec{\theta})$ in these notations is defined as a direct product

$$[\vec{D}_{jk}]_w(\vec{X}, \vec{\theta}) = \hat{D}_w(\vec{X}, \vec{\theta}) \otimes \vec{\mathcal{D}}_{jk}, \quad (20)$$

where $\hat{D}_w(\vec{X}, \vec{\theta})$ is a spinless quantizer operator (3) and $\vec{\mathcal{D}}$ is a 2×2 matrix of four-component vectors

$$\vec{\mathcal{D}} = \left\{ \hat{\mathcal{D}}_{(jk)l} \right\} = \begin{bmatrix} (0, 0, 1, 0) & (1, -i, \frac{-1+i}{2}, \frac{-1+i}{2}) \\ (1, i, -\frac{1+i}{2}, -\frac{1+i}{2}) & (0, 0, 0, 1) \end{bmatrix}, \quad (21)$$

where (jk) are the indexes of 2×2 matrix and $l = 1, 2, 3, 4$ is the index of the component of four-component vector. The functions $w_1(\vec{X}, \vec{\theta})$, $w_2(\vec{X}, \vec{\theta})$, $w_3(\vec{X}, \vec{\theta})$ are the probability distributions of the operator $\hat{X}(\vec{\theta})$ at time t under the conditions that the particle has the value of spin projection equal $1/2$ along q_1 , q_2 , or q_3 directions respectively, and the function $w_4(\vec{X}, \vec{\theta})$ is the probability distribution of this operator under the condition that it has the spin projection $-1/2$ along q_3 direction. Obviously that the two components of the vector $\vec{w}(\vec{X}, \vec{\theta}, t)$ are normalized by the condition

$$\int w_3(\vec{X}, \vec{\theta}, t) d^3X + \int w_4(\vec{X}, \vec{\theta}, t) d^3X = 1. \quad (22)$$

The other two components must be integrable over X^3 and must satisfy the inequalities

$$0 \leq w_j(\vec{X}, \vec{\theta}, t) \leq 1, \quad 0 \leq \int w_j(\vec{X}, \vec{\theta}, t) d^3X \leq 1, \quad j = 1, 2.$$

We see, that the four-component vector $\vec{w}(\vec{X}, \vec{\theta})$ completely define the density matrix $\hat{\rho}$ and consequently it contains all accessible information about the quantum state.

Similarly, for symplectic vector tomography we can write

$$\vec{M}(\vec{X}, \vec{\mu}, \vec{\nu}, t) = \text{Tr}\{\hat{\rho}\vec{U}_M(\vec{X}, \vec{\mu}, \vec{\nu})\}, \quad \hat{\rho}_{jk} = \int [\vec{D}_{jk}]_M(\vec{X}, \vec{\mu}, \vec{\nu}) \vec{M}(\vec{X}, \vec{\mu}, \vec{\nu}, t) d^3X d^3\mu d^3\nu, \quad (23)$$

where dequantizer $\vec{U}_M(\vec{X}, \vec{\mu}, \vec{\nu})$ and quantizer $[\vec{D}_{jk}]_M(\vec{X}, \vec{\mu}, \vec{\nu})$ are defined by the similar formulas (17) and (20) as in the case of optical tomography but the spinless optical dequantizer and quantizer must be replaced with the corresponding symplectic operators (10) and (11)

$$\vec{U}_M(\vec{X}, \vec{\theta}) = \hat{U}_M(\vec{X}, \vec{\theta}) \otimes \vec{\mathcal{U}}, \quad [\vec{D}_{jk}]_M(\vec{X}, \vec{\theta}) = \hat{D}_M(\vec{X}, \vec{\theta}) \otimes \vec{\mathcal{D}}_{jk}. \quad (24)$$

Generalizing equation (5) to the case of spin particles we can write the evolution equation for the vector tomogram in spin optical tomography representation

$$\partial_t \vec{w}_j(\vec{X}, \vec{\theta}, t) = \frac{2}{\hbar} \sum_{k=1}^4 \int \text{Im} \left[\text{Tr} \left\{ \sum_{l,m=1}^2 [\hat{U}_{j(lm)}]_w(\vec{X}, \vec{\theta}) \hat{H} [\hat{D}_{(ml)k}]_w(\vec{X}', \vec{\theta}') \right\} \right] \vec{w}_k(\vec{X}', \vec{\theta}', t) d^n X' d^n \theta', \quad (25)$$

or in spin symplectic tomography representation

$$\begin{aligned} \partial_t \vec{w}_j(\vec{X}, \vec{\mu}, \vec{\nu}, t) = \\ \frac{2}{\hbar} \sum_{k=1}^4 \int \text{Im} \left[\text{Tr} \left\{ \sum_{l,m=1}^2 [\hat{U}_{j(lm)}]_M(\vec{X}, \vec{\mu}, \vec{\nu}) \hat{H} [\hat{D}_{(ml)k}]_M(\vec{X}', \vec{\mu}', \vec{\nu}') \right\} \right] \vec{w}_k(\vec{X}', \vec{\mu}', \vec{\nu}', t) d^n X' d^n \mu' d^n \nu'. \end{aligned} \quad (26)$$

4 Charged spin 1/2 particle in electro-magnetic field

Let's consider the quantum system of a charged spin 1/2 particle with charge e , mass m in electro-magnetic field with potentials $\mathbf{A}(\mathbf{q}, t)$, $\varphi(\mathbf{q}, t)$. As well known, the Hamiltonian of this system has the form

$$\hat{H} = \frac{1}{2m} \left(\hat{\mathbf{p}} - \frac{e}{c} \mathbf{A} \right)^2 + e\varphi - \frac{\varkappa}{s} \hat{\mathbf{s}} \mathbf{H} = \hat{H}_0 - \frac{\varkappa}{s} \hat{\mathbf{s}} \mathbf{H}, \quad (27)$$

where \hat{H}_0 is an independent on spin part of Hamiltonian, $\mathbf{H} = \text{rot} \mathbf{A}$ is a magnetic field, and \varkappa is a magnetic moment of the particle.

Making the transformation of the Pauli equation (4) with the Hamiltonian (27) with the help of transforms (16) and (19) we find the evolution equation of the four-component function $\vec{w}(\vec{X}, \vec{\theta}, t)$ dependent on time t and three component vectors \vec{X} and $\vec{\theta}$

$$\partial_t \vec{w}(\vec{X}, \vec{\theta}, t) = \hat{\mathcal{M}}_w(\vec{X}, \vec{\theta}, t) \vec{w}(\vec{X}, \vec{\theta}, t) + \hat{\mathbf{S}}_w(\vec{X}, \vec{\theta}, t) \vec{w}(\vec{X}, \vec{\theta}, t), \quad (28)$$

where

$$\hat{\mathcal{M}}_w(\vec{X}, \vec{\theta}, t) = \frac{2}{\hbar} \text{Im} \hat{H}_0 \left([\hat{\mathbf{q}}]_w(\vec{X}, \vec{\theta}), [\hat{\mathbf{p}}]_w(\vec{X}, \vec{\theta}), t \right)$$

is an operator depending on two operators of position $[\hat{\mathbf{q}}]_w$ and momentum $[\hat{\mathbf{p}}]_w$ defined by (7) and (8) in the tomographic representation, and $\hat{\mathbf{S}}_w(\vec{X}, \vec{\theta}, t)$ is a 4×4 matrix operator, responsible for the interaction of spin with the magnetic field. With omitted arguments and introduced designations

$$[\hat{A}_j]_w = A_j \left([\hat{\mathbf{q}}]_w(\vec{X}, \vec{\theta}), t \right), \quad \tilde{H}_j = [\hat{H}_j]_w = H_j \left([\hat{\mathbf{q}}]_w(\vec{X}, \vec{\theta}), t \right),$$

$$\left[\nabla_{\mathbf{q}} \hat{\mathbf{A}} \right]_w = \nabla_{\mathbf{q}} \mathbf{A} \left(\mathbf{q} \rightarrow [\hat{\mathbf{q}}]_w(\vec{X}, \vec{\theta}), t \right)$$

the explicit forms of $\hat{\mathcal{M}}_w$ and $\hat{\mathbf{S}}_w$ in general case of timedependent and nonhomogeneous electromagnetic field are written as

$$\begin{aligned} \hat{\mathcal{M}}_w(\vec{X}, \vec{\theta}, t) &= \sum_{n=1}^3 \omega_n \left[\cos^2 \theta_n \frac{\partial}{\partial \theta_n} - \frac{1}{2} \sin 2\theta_n \left\{ 1 + X_n \frac{\partial}{\partial X_n} \right\} \right] + \frac{2e}{\hbar} \text{Im} [\hat{\varphi}]_w \\ &+ \frac{e^2}{mc^2 \hbar} \text{Im} [\hat{\mathbf{A}}]_w^2 - \frac{2e}{mc \hbar} \text{Im} [\hat{\mathbf{A}} \hat{\mathbf{p}}]_w + \frac{e}{mc} \text{Re} [\nabla_{\mathbf{q}} \mathbf{A}]_w, \end{aligned} \quad (29)$$

$$\begin{aligned} [\tilde{S}_{11}]_w &= -2\kappa \text{Im} \tilde{H}_1, \quad [\tilde{S}_{12}]_w = 2\kappa \left\{ -\text{Im} \tilde{H}_2 + \text{Re} \tilde{H}_3 \right\}, \\ [\tilde{S}_{13}]_w &= -\kappa \left\{ 2 \text{Im} \tilde{H}_1 + \text{Re} \tilde{H}_2 - \text{Im} \tilde{H}_2 + \text{Re} \tilde{H}_3 + \text{Im} \tilde{H}_3 \right\}, \quad [\tilde{S}_{14}]_w = \kappa \left\{ \text{Re} \tilde{H}_2 + \text{Im} \tilde{H}_2 - \text{Re} \tilde{H}_3 + \text{Im} \tilde{H}_3 \right\}, \\ [\tilde{S}_{21}]_w &= -2\kappa \left\{ \text{Im} \tilde{H}_1 + \text{Re} \tilde{H}_3 \right\}, \quad [\tilde{S}_{22}]_w = -2\kappa \text{Im} \tilde{H}_2, \\ [\tilde{S}_{23}]_w &= \kappa \left\{ \text{Re} \tilde{H}_1 + \text{Im} \tilde{H}_1 + \text{Re} \tilde{H}_3 - \text{Im} \tilde{H}_3 \right\}, \quad [\tilde{S}_{24}]_w = \kappa \left\{ -\text{Re} \tilde{H}_1 + \text{Im} \tilde{H}_1 + \text{Re} \tilde{H}_3 + \text{Im} \tilde{H}_3 \right\}, \\ [\tilde{S}_{31}]_w &= -2\kappa \left\{ \text{Im} \tilde{H}_1 - \text{Re} \tilde{H}_2 \right\}, \quad [\tilde{S}_{32}]_w = -2\kappa \left\{ \text{Re} \tilde{H}_1 + \text{Im} \tilde{H}_2 \right\}, \\ [\tilde{S}_{33}]_w &= \kappa \left\{ \text{Re} \tilde{H}_1 + \text{Im} \tilde{H}_1 - \text{Re} \tilde{H}_2 + \text{Im} \tilde{H}_2 - 2 \text{Im} \tilde{H}_3 \right\}, \quad [\tilde{S}_{34}]_w = \kappa \left\{ \text{Re} \tilde{H}_1 + \text{Im} \tilde{H}_1 - \text{Re} \tilde{H}_2 + \text{Im} \tilde{H}_2 \right\}, \\ [\tilde{S}_{41}]_w &= -2\kappa \left\{ \text{Im} \tilde{H}_1 + \text{Re} \tilde{H}_2 \right\}, \quad [\tilde{S}_{42}]_w = 2\kappa \left\{ \text{Re} \tilde{H}_1 - \text{Im} \tilde{H}_2 \right\}, \\ [\tilde{S}_{43}]_w &= \kappa \left\{ -\text{Re} \tilde{H}_1 + \text{Im} \tilde{H}_1 + \text{Re} \tilde{H}_2 + \text{Im} \tilde{H}_2 \right\}, \quad [\tilde{S}_{44}]_w = \kappa \left\{ -\text{Re} \tilde{H}_1 + \text{Im} \tilde{H}_1 + \text{Re} \tilde{H}_2 + \text{Im} \tilde{H}_2 + 2 \text{Im} \tilde{H}_3 \right\}. \end{aligned} \quad (30)$$

Making the similar procedure with symplectic vector tomography (23) we can find the evolution equation

$$\partial_t \vec{M}(\vec{X}, \vec{\mu}, \vec{\nu}, t) = \hat{\mathcal{M}}_M(\vec{X}, \vec{\mu}, \vec{\nu}, t) \vec{M}(\vec{X}, \vec{\mu}, \vec{\nu}, t) + \hat{\mathbf{S}}_M(\vec{X}, \vec{\mu}, \vec{\nu}, t) \vec{M}(\vec{X}, \vec{\mu}, \vec{\nu}, t), \quad (31)$$

where operator $\hat{\mathcal{M}}_M(\vec{X}, \vec{\mu}, \vec{\nu}, t)$ corresponds to spinless part \hat{H}_0 of the Hamiltonian (27)

$$\begin{aligned} \hat{\mathcal{M}}_M(\vec{X}, \vec{\mu}, \vec{\nu}, t) &= \frac{2}{\hbar} \text{Im} \hat{H}_0 \left([\hat{\mathbf{p}}]_M(\vec{X}, \vec{\mu}, \vec{\nu}), [\hat{\mathbf{q}}]_M(\vec{X}, \vec{\mu}, \vec{\nu}), t \right) = \vec{\mu} \frac{\partial}{\partial \vec{\nu}} + \frac{2e}{\hbar} \text{Im} [\hat{\varphi}]_M \\ &+ \frac{e^2}{mc^2 \hbar} \text{Im} [\hat{\mathbf{A}}]_M^2 - \frac{2e}{mc \hbar} \text{Im} [\hat{\mathbf{A}} \hat{\mathbf{p}}]_M + \frac{e}{mc} \text{Re} [\nabla_{\mathbf{q}} \mathbf{A}]_M, \end{aligned} \quad (32)$$

where

$$[\hat{A}_j]_M = A_j \left([\hat{\mathbf{q}}]_M(\vec{X}, \vec{\mu}, \vec{\nu}), t \right), \quad [\hat{\varphi}]_M = \varphi \left([\hat{\mathbf{q}}]_M(\vec{X}, \vec{\mu}, \vec{\nu}), t \right),$$

$$[\nabla_{\mathbf{q}}\mathbf{A}]_M = \nabla_{\mathbf{q}}\mathbf{A} \left(\mathbf{q} \rightarrow [\hat{\mathbf{q}}]_M(\vec{X}, \vec{\mu}, \vec{\nu}), t \right),$$

and $[\hat{\mathbf{q}}]_M$, $[\hat{\mathbf{p}}]_M$ are position and momentum operators (13) in the symplectic representation. The 4×4 matrix operator $\hat{\mathbf{S}}_M(\vec{X}, \vec{\mu}, \vec{\nu}, t)$ is defined by the similar formulae (30) where the operators of components of the magnetic field \tilde{H}_j must be replaced with corresponding operators in the symplectic tomography representation $[\hat{H}_j]_M = H_j \left([\hat{\mathbf{q}}]_M(\vec{X}, \vec{\mu}, \vec{\nu}), t \right)$.

5 Evolution equations of charged spin 1/2 particle in Wigner and Husimi representations

Quasiprobability distributions such as Wigner function or Husimi function are powerful tools of description of quantum systems. For spinless particles their definitions from the density matrix can be written as follows

$$W(\mathbf{q}, \mathbf{p}, t) = \text{Tr} \left\{ \hat{\rho}(t) \hat{U}_W(\mathbf{q}, \mathbf{p}) \right\}, \quad (33)$$

$$Q(\mathbf{q}, \mathbf{p}, t) = Q(\vec{\alpha}) = \langle \vec{\alpha} | \hat{\rho}(t) | \vec{\alpha} \rangle = \text{Tr} \left\{ \hat{\rho}(t) \hat{U}_Q(\mathbf{q}, \mathbf{p}) \right\}, \quad (34)$$

with corresponding "dequantizers" having the forms

$$\hat{U}_W(\mathbf{q}, \mathbf{p}) = \frac{1}{(2\pi)^N} \int |\mathbf{q} - \mathbf{u}/2\rangle \exp(-i\mathbf{p}\mathbf{u}/\hbar) \langle \mathbf{q} + \mathbf{u}/2| d^N u, \quad (35)$$

$$\hat{U}_Q(\mathbf{q}, \mathbf{p}) = |\vec{\alpha}\rangle \langle \vec{\alpha}|, \quad \vec{\alpha} = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{m\omega}{\hbar}} \mathbf{q} + \frac{i}{\sqrt{\hbar m\omega}} \mathbf{p} \right), \quad (36)$$

where $|\mathbf{q}\rangle$ is an eigenvalue of the position operator, $|\vec{\alpha}\rangle$ is a coherent state. Inverse maps $W \rightarrow \hat{\rho}$ and $Q \rightarrow \hat{\rho}$ are expressed with corresponding "quantizers"

$$\hat{\rho} = \int \hat{D}_W(\mathbf{q}, \mathbf{p}) W(\mathbf{q}, \mathbf{p}) d^3 q d^3 p = \int \hat{D}_Q(\mathbf{q}, \mathbf{p}) Q(\mathbf{q}, \mathbf{p}) d^3 q d^3 p \quad (37)$$

where quantizers \hat{D}_W and \hat{D}_Q are given by (see [26, 27, 28])

$$\hat{D}_W(\mathbf{q}, \mathbf{p}) = 2^N \int d^N u \exp(2i\mathbf{p}\mathbf{u}/\hbar) |\mathbf{q} + \mathbf{u}\rangle \langle \mathbf{q} - \mathbf{u}|, \quad (38)$$

$$\begin{aligned} \hat{D}_Q(\mathbf{q}, \mathbf{p}) &= \left(\frac{m\omega}{\pi\hbar} \right)^{3/2} \int d^3 x d^3 y \left\{ |\mathbf{x}\rangle \langle \mathbf{y}| \exp \left(\frac{m\omega}{2\hbar} (\mathbf{x} - \mathbf{y})^2 \right) \right. \\ &\times \exp \left[-\frac{m\omega}{\hbar} \left(\mathbf{q} - \frac{\mathbf{x} + \mathbf{y}}{2} \right)^2 - \frac{m\omega}{\hbar} (\mathbf{x} - \mathbf{y})^2 + \frac{i}{\hbar} \mathbf{p}(\mathbf{x} - \mathbf{y}) \right] \\ &\times \left. \prod_{\sigma=1}^3 \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n! 2^n} H_{2n} \left(\sqrt{\frac{m\omega}{\hbar}} q_{\sigma} - \frac{m\omega}{2\hbar} (x_{\sigma} + y_{\sigma})^2 \right) \right] \right\} \end{aligned} \quad (39)$$

Such definitions provide that the quasidistributions in spinless case are real functions contrary to density matrices, whose nondiagonal elements may be complex. If the particle has spin, the density matrix $\hat{\rho}_{jk}$ additionally depend on spin indexes, and usually in literature many authors make generalizations of definitions (33), (34) handling the trace operations as a partial trace over all of the variables excepting spin indexes. In such definitions Wigner function $W_{jk}(\mathbf{q}, \mathbf{p}, t)$ and Husimi function $Q_{jk}(\mathbf{q}, \mathbf{p}, t)$ become $(2s+1) \times (2s+1)$ matrices dependent on position and momentum. But their nondiagonal elements over the spin indexes are not surely real. So, the main advantage of such quasidistributions with respect to density matrix disappears.

To decide this problem let us expand our approach to the quasidistributions and define four-component Wigner $\vec{W}(\mathbf{q}, \mathbf{p}, t)$ and Husimi $\vec{Q}(\mathbf{q}, \mathbf{p}, t)$ vector-functions as follows

$$\vec{W}(\mathbf{q}, \mathbf{p}, t) = \text{Tr} \left\{ \hat{\rho}(t) \vec{\hat{U}}_W(\mathbf{q}, \mathbf{p}) \right\}, \quad (40)$$

$$\vec{Q}(\mathbf{q}, \mathbf{p}, t) = \text{Tr} \left\{ \hat{\rho}(t) \vec{\hat{U}}_Q(\mathbf{q}, \mathbf{p}) \right\}, \quad (41)$$

where dequantizers $\vec{\hat{U}}_W$ and $\vec{\hat{U}}_Q$ are defined by the same formulae as (17) with replacement of $\vec{\hat{U}}(\vec{X}, \vec{\theta})$ by $\hat{U}_W(\mathbf{q}, \mathbf{p})$ or $\hat{U}_Q(\mathbf{q}, \mathbf{p})$

$$\vec{\hat{U}}_W(\vec{X}, \vec{\theta}) = \hat{U}_W(\vec{X}, \vec{\theta}) \otimes \vec{\hat{U}}, \quad \vec{\hat{U}}_Q(\vec{X}, \vec{\theta}) = \hat{U}_Q(\vec{X}, \vec{\theta}) \otimes \vec{\hat{U}}. \quad (42)$$

Such definitions guarantee that all of the components of $\vec{W}(\mathbf{q}, \mathbf{p}, t)$ and $\vec{Q}(\mathbf{q}, \mathbf{p}, t)$ are real, more over, all of the components of $\vec{Q}(\mathbf{q}, \mathbf{p}, t)$ are nonnegative. Here $W_j(\mathbf{q}, \mathbf{p}, t)$ and $Q_j(\mathbf{q}, \mathbf{p}, t)$ are components of Wigner and Husimi vector quasiprobability corresponding definite spin projection along q_1 , q_2 , or q_3 direction.

Inverse mappings of (40) and (41) are obvious

$$\hat{\rho}_{jk}(t) = \int [\vec{\hat{D}}_{jk}]_W(\mathbf{q}, \mathbf{p}) \vec{W}(\mathbf{q}, \mathbf{p}, t) d^3q d^3p = \int [\vec{\hat{D}}_{jk}]_Q(\mathbf{q}, \mathbf{p}) \vec{Q}(\mathbf{q}, \mathbf{p}, t) d^3q d^3p, \quad (43)$$

where quantizers $[\vec{\hat{D}}_{jk}]_W$ and $[\vec{\hat{D}}_{jk}]_Q$ are defined by the similar formulas as in the case of optical tomography (17), (20) but the spinless optical quantizers must be replaced with the corresponding operators (38) and (39)

$$[\vec{\hat{D}}_{jk}]_W(\mathbf{q}, \mathbf{p}) = \hat{D}_W(\mathbf{q}, \mathbf{p}) \otimes \vec{\hat{D}}_{jk}, \quad [\vec{\hat{D}}_{jk}]_Q(\mathbf{q}, \mathbf{p}) = \hat{D}_Q(\mathbf{q}, \mathbf{p}) \otimes \vec{\hat{D}}_{jk}. \quad (44)$$

Let us give the expression for Wigner function in terms of the Husimi function

$$\vec{W}(\mathbf{q}, \mathbf{p}) = \exp \left(-\frac{\hbar}{4m\omega} \Delta_{\mathbf{q}} - \frac{1}{4m\omega\hbar} \Delta_{\mathbf{p}} \right) \vec{Q}(\mathbf{q}, \mathbf{p}), \quad (45)$$

where $\Delta_{\mathbf{q}}$ and $\Delta_{\mathbf{p}}$ are Laplace operators in 3D spaces $\{q_j\}$ and $\{p_j\}$. This formula is a trivial generalization of the corresponding formula [29] for spinless W and Q .

Likewise we can introduce the vector Glauber-Sudarshan P-function [30, 31] $\vec{P}(\vec{\alpha}, t)$

$$\vec{P}(\vec{\alpha}, t) = \text{Tr} \left\{ \hat{\rho}(t) \vec{U}_P(\vec{\alpha}) \right\}, \quad \hat{\rho}(t) = \int [\vec{D}]_P(\vec{\alpha}) \vec{P}(\vec{\alpha}, t) d^{2n}\alpha,$$

where

$$\vec{U}_P(\vec{\alpha}) = \left(\frac{e^{|\vec{\alpha}|^2}}{\pi^n} \int |\vec{\beta}\rangle \langle \vec{\beta}| e^{|\vec{\beta}|^2 - \vec{\beta}^* \vec{\alpha} + \vec{\beta} \vec{\alpha}^*} d^{2n}\beta \right) \otimes \vec{U}, \quad [\vec{D}]_P(\vec{\alpha}) = |\alpha\rangle \langle \alpha| \otimes \vec{D}.$$

In previous sections we have found the evolution equation of charged spin 1/2 particle in electromagnetic field in optical and symplectic tomographic representations. Making similar calculation we can obtain such evolution equation for our vector Wigner function, which will be a generalization of the Moyal equation [32]

$$\begin{aligned} \frac{\partial}{\partial t} \vec{W}(\mathbf{q}, \mathbf{p}, t) &= \left[-\frac{\mathbf{p}}{m} \frac{\partial}{\partial \mathbf{q}} + \frac{2e}{\hbar} \text{Im} \varphi \left(\mathbf{q} + \frac{i\hbar}{2} \frac{\partial}{\partial \mathbf{p}}, t \right) + \frac{e^2}{mc^2 \hbar} \text{Im} \mathbf{A}^2 \left(\mathbf{q} + \frac{i\hbar}{2} \frac{\partial}{\partial \mathbf{p}}, t \right) \right. \\ &+ \left. -\frac{2e}{mc\hbar} \text{Im} \left\{ \mathbf{A} \left(\mathbf{q} + \frac{i\hbar}{2} \frac{\partial}{\partial \mathbf{p}}, t \right) \left(\mathbf{p} - \frac{i\hbar}{2} \frac{\partial}{\partial \mathbf{q}} \right) \right\} \right. \\ &+ \left. \frac{e}{mc} \text{Re} \nabla_{\mathbf{q}} \mathbf{A} \left(\mathbf{q} \rightarrow \mathbf{q} + \frac{i\hbar}{2} \frac{\partial}{\partial \mathbf{p}}, t \right) + \hat{\mathbf{S}}_W(\mathbf{q}, \mathbf{p}, t) \right] \vec{W}(\mathbf{q}, \mathbf{p}, t), \end{aligned} \quad (46)$$

where 4×4 matrix operator $\hat{\mathbf{S}}_W(\mathbf{q}, \mathbf{p}, t)$ is defined by the same formulae (30) where the operators of components of the magnetic field \tilde{H}_j must be replaced with corresponding operators in the Wigner representation $H_j \left(\mathbf{q} + \frac{i\hbar}{2} \frac{\partial}{\partial \mathbf{p}}, t \right)$.

The corresponding equation for Husimi function is obtained from (46) with the help of expression (45) (see also [28]). For simplicity we choose the system of measurements so that $m = \omega = \hbar = 1$

$$\begin{aligned} \frac{\partial}{\partial t} \vec{Q}(\mathbf{q}, \mathbf{p}, t) &= \left[-\mathbf{p} \frac{\partial}{\partial \mathbf{q}} - \frac{1}{2} \frac{\partial}{\partial \mathbf{q}} \frac{\partial}{\partial \mathbf{p}} + \frac{2e}{\hbar} \text{Im} \varphi \left(\mathbf{q} + \frac{1}{2} \frac{\partial}{\partial \mathbf{q}} + \frac{i}{2} \frac{\partial}{\partial \mathbf{p}}, t \right) \right. \\ &+ \frac{e^2}{c^2} \text{Im} \mathbf{A}^2 \left(\mathbf{q} + \frac{1}{2} \frac{\partial}{\partial \mathbf{q}} + \frac{i}{2} \frac{\partial}{\partial \mathbf{p}}, t \right) \\ &- \frac{2e}{c} \text{Im} \left\{ \mathbf{A} \left(\mathbf{q} + \frac{1}{2} \frac{\partial}{\partial \mathbf{q}} + \frac{i}{2} \frac{\partial}{\partial \mathbf{p}}, t \right) \left(\mathbf{p} + \frac{1}{2} \frac{\partial}{\partial \mathbf{p}} - \frac{i}{2} \frac{\partial}{\partial \mathbf{q}} \right) \right\} \\ &+ \left. \frac{e}{c} \text{Re} \nabla_{\mathbf{q}} \mathbf{A} \left(\mathbf{q} \rightarrow \mathbf{q} + \frac{1}{2} \frac{\partial}{\partial \mathbf{q}} + \frac{i}{2} \frac{\partial}{\partial \mathbf{p}}, t \right) + \hat{\mathbf{S}}_Q(\mathbf{q}, \mathbf{p}, t) \right] \vec{Q}(\mathbf{q}, \mathbf{p}, t), \end{aligned} \quad (47)$$

where 4×4 matrix operator $\hat{\mathbf{S}}_Q(\mathbf{q}, \mathbf{p}, t)$ is defined by (30) in which components of the magnetic field \tilde{H}_j are replaced with $H_j \left(\mathbf{q} + \frac{1}{2} \frac{\partial}{\partial \mathbf{q}} + \frac{i}{2} \frac{\partial}{\partial \mathbf{p}}, t \right)$.

6 Charged spin 1/2 in homogeneous magnetic field

Choose the vector and scalar potentials as follows (Landau gauge)

$$\mathbf{A} = (-q_2 H, 0, 0), \quad \varphi = 0, \quad \mathbf{H} = (0, 0, H), \quad (48)$$

suppose $\omega_1 = \omega_2 = \omega_3 = |\omega|$, where $\omega = \frac{eH}{mc}$, and choose the system of measurements so that $|\omega| = m = \hbar = 1$. In these units $\omega = +1$ when the charge is positive and $\omega = -1$ when it is negative.

Also denote the frequency of spin rotation as $\omega_0 = \frac{\kappa H}{\hbar}$. For electrons $\omega_0 = \omega$, but for other particles these two frequencies may be different.

Then the Hamiltonian will have the form

$$\hat{H} = \frac{1}{2}\hat{p}_1^2 + \frac{1}{2}\hat{p}_2^2 + \frac{1}{2}\hat{p}_3^2 + \frac{1}{2}\hat{q}_2^2 + \omega\hat{p}_1\hat{q}_2 - \frac{\omega_0}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (49)$$

Using the general formulae (28) – (30) we find the evolution equation

$$\begin{aligned} \frac{\partial}{\partial t} \vec{w}(\vec{X}, \vec{\theta}, t) &= \left[\sum_{\sigma=1,3} \left(\cos^2 \theta_\sigma \frac{\partial}{\partial \theta_\sigma} - \frac{1}{2} \sin 2\theta_\sigma \left\{ 1 + X_\sigma \frac{\partial}{\partial X_\sigma} \right\} \right) + \frac{\partial}{\partial \theta_2} \right. \\ &\quad - \omega \left(\cos \theta_1 \left[\frac{\partial}{\partial X_1} \right]^{-1} - X_1 \sin \theta_1 \right) \sin \theta_2 \frac{\partial}{\partial X_2} \\ &\quad \left. - \omega \left(\sin \theta_2 \left[\frac{\partial}{\partial X_2} \right]^{-1} + X_2 \cos \theta_2 \right) \cos \theta_1 \frac{\partial}{\partial X_1} + \hat{\mathbf{S}}_w \right] \vec{w}(\vec{X}, \vec{\theta}, t), \end{aligned} \quad (50)$$

where upper signs correspond to positive charge and lower signs correspond to negative charge, and the matrix $\hat{\mathbf{S}}_w$ is given by the expression

$$\hat{\mathbf{S}}_w = \omega_0 \begin{bmatrix} 0 & 1 & -1/2 & -1/2 \\ -1 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (51)$$

or in symplectic tomography representation

$$\begin{aligned} \frac{\partial}{\partial t} \vec{M}(\vec{X}, \vec{\mu}, \vec{\nu}) &= \left[\vec{\mu} \frac{\partial}{\partial \vec{\nu}} - \omega \nu_2 \left[\frac{\partial}{\partial X_1} \right]^{-1} \frac{\partial}{\partial X_2} \frac{\partial}{\partial \nu_1} \right. \\ &\quad \left. + \omega \mu_1 \left[\frac{\partial}{\partial X_2} \right]^{-1} \frac{\partial}{\partial X_1} \frac{\partial}{\partial \nu_2} + \hat{\mathbf{S}}_M \right] \vec{M}(\vec{X}, \vec{\mu}, \vec{\nu}). \end{aligned} \quad (52)$$

In Wigner representation this equation has the form

$$\frac{\partial}{\partial t} \vec{W}(\mathbf{q}, \mathbf{p}, t) = \left[-\mathbf{p} \frac{\partial}{\partial \mathbf{q}} + q_2 \frac{\partial}{\partial p_2} + \omega p_1 \frac{\partial}{\partial p_2} - \omega q_2 \frac{\partial}{\partial q_1} + \hat{\mathbf{S}}_w \right] \vec{W}(\mathbf{q}, \mathbf{p}, t). \quad (53)$$

It is obvious that for homogeneous magnetic field $\hat{\mathbf{S}}_M = \hat{\mathbf{S}}_w = \hat{\mathbf{S}}_W = \hat{\mathbf{S}}_Q$.

If we integrate equation (50) over d^3X and introduce the notation $\int \vec{w}(\vec{X}, \vec{\theta}, t) d^3X = \vec{P}(t)$, then equation (50) take the form

$$\partial_t \vec{P}(t) = \hat{\mathbf{S}}_w \vec{P}(t). \quad (54)$$

This equation corresponds to the case when we are interesting in only spin dynamic. After some calculations we get the propagator of this equation and, consequently, the solution $\vec{P}(t) = \hat{\Pi}_s(t)\vec{P}(0)$ for arbitrary initial condition $\vec{P}(0)$

$$\hat{\Pi}_s(t) = \begin{bmatrix} \cos \omega_0 t & \sin \omega_0 t & (1 - \cos \omega_0 t - \sin \omega_0 t)/2 & (1 - \cos \omega_0 t - \sin \omega_0 t)/2 \\ -\sin \omega_0 t & \cos \omega_0 t & (1 - \cos \omega_0 t + \sin \omega_0 t)/2 & (1 - \cos \omega_0 t + \sin \omega_0 t)/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (55)$$

We can see, that the probability of finding the particle in the state with the spin projection $\pm 1/2$ along q_3 direction remains constant during evolution.

The spinless part of the propagator for eq. (50) or (52) corresponds to the free motion along q_3 direction and to the evolution of ordinary quadratic system with respect to q_1 and q_2 degrees of freedom. For free motion the propagator was found in [33] for optical tomography and in [20] for symplectic one

$$[\Pi_f]_w(X_3, \theta_3, X'_3, \theta'_3, t) = \delta(X_3 \cos \theta'_3 - X'_3 \cos \theta_3) \delta(\cos \theta'_3(t + \tan \theta_3) - \sin \theta'_3), \quad (56)$$

$$[\Pi_f]_M(X_3, \mu_3, \nu_3, X'_3, \mu'_3, \nu'_3) = \delta(X_3 - X'_3) \delta(\nu'_3 - \nu_3 - \mu_3 t) \delta(\mu_3 - \mu'_3). \quad (57)$$

Free motion propagator for the Wigner function, obviously, equals

$$[\Pi_f]_W(q_3, p_3, q'_3, p'_3, t) = \delta\left(q_3 - q'_3 - \frac{p'_3}{m}t\right) \delta(p_3 - p'_3). \quad (58)$$

For quadratic subsystem the propagator can be found by the method of motion integrals, or it can be obtained from the known propagator for the wave function by means of transformation to the corresponding representation (see [33, 34, 35]). After some calculations we have

$$\begin{aligned} & [\Pi_{12}]_w(X_1, X_2, \theta_1, \theta_2, X'_1, X'_2, \theta'_1, \theta'_2, t) = \\ & = \delta \left\{ \cos \theta_1 \tan \left(\frac{\omega t}{2} \right) [2 \cos \theta'_1 - \omega \sin \theta'_2] + \sin(\theta_1 - \theta'_1) \right\} \\ & \times \delta \left\{ \cos \theta_2 \tan \left(\frac{\omega t}{2} \right) \left[2 \cos \theta'_2 + \omega \frac{\sin(\theta_1 - \theta'_1)}{\cos \theta_1 \sin \theta_1} \right] + \sin(\theta_2 - \theta'_2) + \omega \frac{\sin \theta_2 \sin(\theta_1 - \theta'_1)}{\cos \theta_1 \sin \theta_1} \right\} \\ & \times \delta \left\{ X_1 \frac{\cos \theta'_1}{\cos \theta_1} + \omega X_2 \frac{\sin(\theta_1 - \theta'_1)}{\cos \theta_2 \cos \theta_1 \sin \theta_1} - X'_1 \right\} \delta \left\{ X_2 \frac{\cos \theta'_2}{\cos \theta_2} - X'_2 \right\}, \end{aligned} \quad (59)$$

or in symplectic representation

$$\begin{aligned}
[\Pi_{12}]_M(X_1, X_2, \mu_1, \mu_2, \nu_1, \nu_2, X'_1, X'_2, \mu'_1, \mu'_2, \nu'_1, \nu'_2, t) = \\
= (2\pi)^{-2} \exp \left\{ i \left[-X_1 \frac{\mu'_1}{\mu_1} - X_2 \frac{\mu'_2}{\mu_2} + X'_1 + X'_2 - \frac{\omega X_2}{\mu_2} \left(\frac{\mu'_1}{\mu_1} - \frac{\nu'_1}{\nu_1} \right) \right] \right\} \\
\times \delta \left\{ \mu_1 \tan \left(\frac{\omega t}{2} \right) (2\mu'_1 - \omega \nu'_2) - \mu_1 \nu'_1 + \mu'_1 \nu_1 \right\} \\
\times \delta \left\{ \mu_2 \tan \left(\frac{\omega t}{2} \right) \left[2\mu'_2 + \omega \left(\frac{\mu'_1}{\mu_1} - \frac{\nu'_1}{\nu_1} \right) \right] + \omega \nu_2 \left(\frac{\mu'_1}{\mu_1} - \frac{\nu'_1}{\nu_1} \right) - \mu_2 \nu'_2 + \mu'_2 \nu_2 \right\}. \quad (60)
\end{aligned}$$

In Wigner representation this propagator has the form

$$\begin{aligned}
[\Pi_{12}]_W(q_1, q_2, p_1, p_2, q'_1, q'_2, p'_1, p'_2, t) = \frac{m^2 \omega^2}{4\hbar^2 \sin^2(\omega t/2)} \\
\times \delta \left\{ \frac{m\omega}{2\hbar} \cot \left(\frac{\omega t}{2} \right) (q_1 - q'_1) - \frac{m\omega}{2\hbar} (q_2 - q'_2) - p'_1 \right\} \\
\times \delta \left\{ \frac{m\omega}{2\hbar} \cot \left(\frac{\omega t}{2} \right) (q_2 - q'_2) + \frac{m\omega}{2\hbar} (q_1 - q'_1) - p'_2 \right\} \\
\times \delta \left\{ \frac{m\omega}{\hbar} (q_1 - q'_1) + p_2 - p'_2 \right\} \delta \{ p_1 - p'_1 \}. \quad (61)
\end{aligned}$$

The total propagator of eq.(50) equals to the product of corresponding propagators (55), (56), and (59)

$$\hat{\Pi}_w(\vec{X}, \vec{\theta}, \vec{X}', \vec{\theta}', t) = \hat{\Pi}_s \otimes [\Pi_f]_w \otimes [\Pi_{12}]_w. \quad (62)$$

Reciprocally for symplectic or Wigner representation we have

$$\hat{\Pi}_M(\vec{X}, \vec{\mu}, \vec{\nu}, \vec{X}', \vec{\mu}', \vec{\nu}', t) = \hat{\Pi}_s \otimes [\Pi_f]_M \otimes [\Pi_{12}]_M, \quad (63)$$

$$\hat{\Pi}_W(\mathbf{q}, \mathbf{p}, \mathbf{q}', \mathbf{p}', t) = \hat{\Pi}_s \otimes [\Pi_f]_W \otimes [\Pi_{12}]_W. \quad (64)$$

Let us average the evolution of the system over free motion along q_3 direction, i.e. integrate the evolution equation and initial condition over X_3 and consider an initial condition which is the entangled superposition of lower Landau levels of electron in Landau gauge (48) (the charge is negative and in our notations $\omega = \omega_0 = -1$). For this we introduce two pairs of creation and annihilation operators \hat{a} , \hat{a}^\dagger and \hat{b} , \hat{b}^\dagger (see [36, 37])

$$\hat{a} = (\hat{p}_1 - \hat{q}_2 - i\hat{p}_2)/\sqrt{2}, \quad \hat{a}^\dagger = (\hat{p}_1 - \hat{q}_2 + i\hat{p}_2)/\sqrt{2}, \quad (65)$$

$$\hat{b} = (\hat{q}_1 - \hat{p}_2 + i\hat{p}_1)/\sqrt{2}, \quad \hat{b}^\dagger = (\hat{q}_1 - \hat{p}_2 - i\hat{p}_1)/\sqrt{2}. \quad (66)$$

These operators have the following properties

$$[\hat{a}, \hat{a}^\dagger] = 1, \quad [\hat{b}, \hat{b}^\dagger] = 1, \quad [\hat{a}, \hat{b}] = [\hat{a}, \hat{b}^\dagger] = 0,$$

and Hamiltonian (49) without q_3 degree of freedom is expressed in terms of \hat{a} and \hat{a}^\dagger as

$$\hat{H} = \hat{a}^\dagger \hat{a} + \frac{1}{2} + \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Landau levels correspond to the states with wave functions

$$|nm\rangle = \frac{(\hat{a}^\dagger)^n (\hat{b}^\dagger)^m}{\sqrt{n!m!}} |00\rangle,$$

where $|00\rangle$ is a vacuum state of the system

$$\hat{a}|00\rangle = 0, \quad \hat{b}|00\rangle = 0, \quad \langle 00|00\rangle = 1,$$

$$\langle q_1, q_2 | 00 \rangle = \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{q_1^2}{4} - \frac{q_2^2}{4} + i \frac{q_1 q_2}{2} \right).$$

The first excited state $|10\rangle$ has the wave function

$$\langle q_1, q_2 | 10 \rangle = \langle q_1, q_2 | \hat{a}^\dagger | 00 \rangle = \frac{iq_1 - q_2}{2\sqrt{\pi}} \exp \left(-\frac{q_1^2}{4} - \frac{q_2^2}{4} + i \frac{q_1 q_2}{2} \right).$$

Consider an initial condition which is the entangled superposition of lower Landau levels

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}} (|00\rangle \otimes |-1/2\rangle + |10\rangle \otimes |1/2\rangle).$$

It corresponds to our vector optical tomogram

$$\vec{w}(X_1, X_2, \theta_1, \theta_2, 0) = \frac{1}{4} \begin{pmatrix} w_{0000} + w_{1010} + 2\text{Re } w_{0010} \\ w_{0000} + w_{1010} + 2\text{Im } w_{0010} \\ 2w_{1010} \\ 2w_{0000} \end{pmatrix},$$

where we introduce the designation

$$w_{nmn'm'} = \langle X_1, X_2, \theta_1, \theta_2 | nm \rangle \langle n'm' | X_1, X_2, \theta_1, \theta_2 \rangle.$$

It is easy to see that the corresponding solution of evolution equation will be

$$\vec{w}(X_1, X_2, \theta_1, \theta_2, t) = \frac{1}{4} \begin{pmatrix} w_{0000} + w_{1010} + 2\text{Re}\{w_{0010} \exp(i2t)\} \\ w_{0000} + w_{1010} + 2\text{Im}\{w_{0010} \exp(i2t)\} \\ 2w_{1010} \\ 2w_{0000} \end{pmatrix}.$$

Double frequency here is the result of interference of simultaneous spin rotation and cyclotron quantum motion in the plane perpendicular to the magnetic field.

7 Linear harmonic oscillator with spin

As another example we consider a system with following Hamiltonian:

$$\hat{H} = \frac{1}{2}(p^2 + q^2) + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It could describe one vibrational degree of a trapped electron plus its spin [38]. The measurability of tomograms in this system was investigated in [39].

The evolution equation of the vector tomogram for this system has the simple form

$$\frac{\partial}{\partial t} \vec{w}(X, \theta, t) = \frac{\partial}{\partial \theta} \vec{w}(X, \theta, t) + \hat{\mathbf{S}}_w \vec{w}(X, \theta, t), \quad (67)$$

where $\hat{\mathbf{S}}_w$ is defined by (51) with $\omega_0 = -2$, and the propagator for this equation be

$$\hat{\Pi}(X, \theta, t) = \delta(\theta - t - \theta') \otimes \hat{\Pi}_s.$$

If we take an initial entangled state

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |-1/2\rangle + |1\rangle \otimes |1/2\rangle)$$

with initial vector optical tomogram

$$\vec{w}(X, \theta, 0) = \frac{1}{4} \begin{pmatrix} w_{00} + w_{11} + 2\text{Re } w_{01} \\ w_{00} + w_{11} + 2\text{Im } w_{01} \\ 2w_{11} \\ 2w_{00} \end{pmatrix},$$

where

$$\begin{aligned} w_{00} &= \langle X, \theta | 0 \rangle \langle 0 | X, \theta \rangle = \frac{1}{\sqrt{\pi}} e^{-X^2}, \\ w_{11} &= \langle X, \theta | 1 \rangle \langle 1 | X, \theta \rangle = \frac{2}{\sqrt{\pi}} X^2 e^{-X^2}, \\ w_{01} &= \langle X, \theta | 1 \rangle \langle 0 | X, \theta \rangle = \frac{\sqrt{2}}{\sqrt{\pi}} X e^{i\theta} e^{-X^2}, \end{aligned}$$

then, the solution of equation (67) be

$$\vec{w}(X, \theta, t) = \frac{1}{4} \begin{pmatrix} w_{00} + w_{11} + 2\text{Re}\{w_{01} \exp(i3t)\} \\ w_{00} + w_{11} + 2\text{Im}\{w_{01} \exp(i3t)\} \\ 2w_{11} \\ 2w_{00} \end{pmatrix},$$

and we can see again the addition of two frequencies: the harmonic oscillator frequency and the frequency of spin rotation.

8 Conclusion

To resume we point out the main results of our paper. We suggested to describe the state of charged spin 1/2 particle by a new four-component positive vector of joint probability distributions, that is the vector optical tomogram. Such approach of construction of positive vector-portrait of quantum state eliminates the redundancy, which is the main difficulty of schemes proposed by another authors. Reciprocally we introduce the vector symplectic tomogram and vector quasidistributions $\vec{W}(\mathbf{q}, \mathbf{p})$, $\vec{Q}(\mathbf{q}, \mathbf{p})$, $\vec{P}(\vec{\alpha})$.

We obtained the evolution equations for such vector optical and symplectic tomograms and vector quasidistributions for arbitrary Hamiltonian. We considered in proposed representations the quantum system of charged spin 1/2 particle in arbitrary electro-magnetic field and obtained evolution equations, which are analogs of Pauli equation in appropriate representations.

As an example we found the propagator of evolution equation in the case of homogeneous and stationary magnetic field in Landau gauge, we considered the evolution of initial entangled superposition of lower Landau levels in the vector optical representation and illustrated the addition of the frequency of simultaneous spin rotation and the frequency of cyclotron quantum motion in the plane perpendicular to the magnetic field.

Also as an example we considered the system of linear quantum oscillator with spin in vector optical tomography representation and studied the evolution of initial entangled superposition of two lower Fock states and spin-up spin-down states.

A possible disadvantage of the approach proposed is a relatively complicated evolution equations, but this is the price one ought to pay for the possibility of describing quantum objects in term of classical probabilities. In addition, the equations obtained in this paper are much more easier than in the previous attempt [24] of description of evolution of spin particles in terms of probabilities.

The generalization of the results of this paper to the higher spin particles will be given in further publications.

References

- [1] J. Bertrand and P. Bertrand, *Found. Phys.*, **17**, 397 (1987).
- [2] K. Vogel and H. Risken, *Phys. Rev. A*, **40**, 2847 (1989).
- [3] S. Mancini, V. I. Man'ko and P. Tombesi, *Phys. Lett. A*, **213**, 1 (1996).
- [4] S. Mancini, V. I. Man'ko, and P. Tombesi, *Found. Phys.*, **27**, 801 (1997).
- [5] V. V. Dodonov, V. I. Man'ko, *Phys. Lett. A*, **229**, 335 (1997).

- [6] J.-P. Amiet, S. Weigert, *J. Phys. A: Math. Gen.*, **31**, L543 (1998).
- [7] A. Ibort, V. I. Man'ko, G. Marmo, A. Simoni, and F. Ventriglia, *Phys. Scr.*, **79**, 065013 (2009).
- [8] M. A. Man'ko and V. I. Man'ko, *Found. Phys.*, **41**, 330 (2011).
- [9] A. I. Lvovsky and M. G. Raymer, *Rev. Mod. Phys.*, **81**, 299 (2009).
- [10] R. G. Newton and B. Young, *Ann. Phys.*, **49**, 393 (1968).
- [11] S. Weigert, *Phys. Rev. A*, **45**, 7688 (1992).
- [12] J.-P. Amiet and S. Weigert, *J. Phys. A: Math. Gen.*, **31**, L543 (1998).
- [13] J.-P. Amiet and S. Weigert, *J. Phys. A: Math. Gen.*, **32**, 2777 (1999).
- [14] J.-P. Amiet and S. Weigert, *J. Opt. B: Quantum Semiclass. Opt.*, **1**, L5 (1999).
- [15] J.-P. Amiet and S. Weigert, *J. Phys. A: Math. Gen.*, **32**, L269 (1999).
- [16] J.-P. Amiet and S. Weigert, *J. Opt. B: Quantum Semiclass. Opt.*, **2**, 118 (2000).
- [17] S. Heiss and S. Weigert, *Phys. Rev. A*, **63**, 012105 (2000).
- [18] S. N. Filippov, V. I. Man'ko, *J. Russ. Laser Res.*, **31**, 32 (2010).
- [19] V. I. Man'ko, G. Marmo, E. C. G. Sudarshan, F. Zaccaria, *Phys. Lett. A*, **327**, 353 (2004).
- [20] O. V. Man'ko and V. I. Man'ko, *J. Russ. Laser Res.*, **18**, 407 (1997).
- [21] Ya. A. Korennoy, V. I. Man'ko, *J. Russ. Laser Res.*, **32**, 74 (2011).
- [22] F. Lizzi, P. Vitale, *SIGMA*, **10**, 086 (2014).
- [23] Ya. A. Korennoy, V. I. Man'ko, *J. Russ. Laser Res.* **32**, 338 (2011).
- [24] S. Mancini, Olga V. Man'ko, V. I. Man'ko, P. Tombesi, *J. Phys. A: Math. Gen.*, **34**, 3461 (2001).
- [25] G. G. Amosov, Ya. A. Korennoy, V. I. Man'ko, *Phys. Rev. A*, **85**, 052119 (2012) .
- [26] E. Wigner, *Phys. Rev.*, **40**, 749, (1932).
- [27] K. Husimi, *Proc. Phys.-Math. Soc. Japan*, **22**, 264-314 (1940).
- [28] S. S. Mizrahi, *Physica A*, **127**, 241 (1984).
- [29] D. M. Davidović, D. Lalović, *J. Phys. A*, **26**, 5099 (1993).

- [30] R. J. Glauber, *Phys. Rev. Lett.*, **10**, 84 (1963).
- [31] E. C. G. Sudarshan, *Phys. Rev.*, **10**, 277 (1963).
- [32] J.E. Moyal, *Proc. Cambrige Philos. Soc.*, **45**, 99 (1949).
- [33] Ya. A. Korennoy, V. I. Man'ko, *J. Russ. Laser Res.*, **32**, 153 (2011).
- [34] V. V. Dodonov and V. I. Man'ko, *Invariants and the Evolution of Nonstationary Quantum Systems*, *Proceedings of the Lebedev Physical Institute*, Nova Science, New York (1989), Vol. 183.
- [35] V. I. Manko, L. Rosa, P. Vitale *Phys. Rev. A*, **57**, 3291 (1998).
- [36] L. D. Landau, E. M. Lifshitz, *Quantum mechanics*, (Pergamo, Oxford, 1997).
- [37] I. A. Malkin, V. I. Man'ko, *Zh. Eksp. Teor. Fiz.* **55**, 1014 (1968).
- [38] L. S. Brown, G. Gabrielse, *Rev. Mod. Phys.* **58**, 233 (1986).
- [39] M. Massini, M. Fortunato, S. Mancini, P. Tombesi, *Phys. Rev. A* **62**, 041401(R) (2000).